(54) Title: DIAGNOSTIC SIGNAL PROCESSING METHOD AND SYSTEM

ACTUAL AND CALCULATED CUMULATIVE SPECTRAL DENSITY FUNCTION
FROM DIGITIZED DATA M=200
TOTAL AMPLITUDE CUMULATIVE DISTRIBUTION BASED ON A SPECTRAL
DENSITY FUNCTION USING 7 SEGMENTS WITH 200 MEASUREMENTS
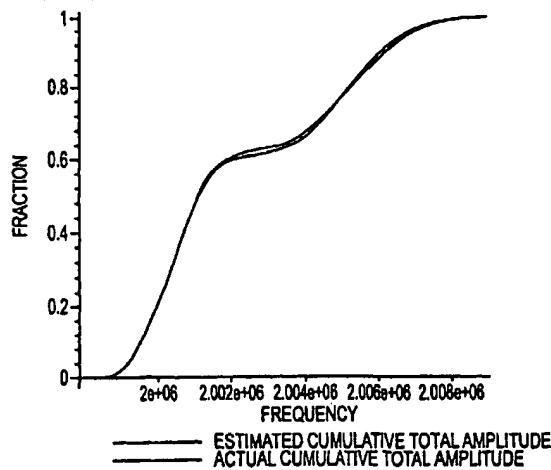
(57) Abstract: A signal processing utility is disclosed involving time-to-frequency domain transforms for applications including medical diagnostic signal processing. Such transforms can be used to define a continuous spectral density function or other spectral density function including non-zero values at irregularly spaced frequency intervals. The invention thereby enables more accurate representation of certain real spectra and reduced spectral broadening. The utility also accounts for digitization errors associated with analog-to-digital conversion. The invention has particular advantages with respect to medical contexts where the received signal has a changing spectral content as a result of interaction of an interrogating signal with moving physiological material such as blood flowing through an artery.

DIAGNOSTIC SIGNAL PROCESSING METHOD AND SYSTEM

FIELD OF THE INVENTION

The present invention relates generally to signal processing applications and, in particular, to signal processing applications involving time-to-frequency domain transforms where it is useful to define a spectrum including frequency information for frequencies other than a fundamental

5   frequency and harmonies thereof, or where it is useful to generate a spectrum based on a portion of the time-based signal representing less than a period of a signal component. The invention has particular application to methods wherein a received signal is decomposed or otherwise processed in a manner that provides information regarding properties of a material modulating or otherwise producing the received signal. The received signal may emanate from tissue interacting with an

10   interrogating signal that was introduced into an organism and was modified from the interrogating signal by one or more influences of the organism's material, e.g., organs, tissue, fluids or other structure.

BACKGROUND OF THE INVENTION

15   Noninvasive diagnoses using interrogating signals are frequently employed to ascertain the condition of tissue and functional parameters of organs. Such interrogating signals may use, for example, electromagnetic or sonic, including ultrasonic, energy. One example is transmitting ultrasonic waves into an organism, such as a human being, and receiving a return signal that is a modified version of the transmitted interrogating signal, with the modifications caused by the

20   properties of one or more tissue types. The received signal is measured and processed to evaluate the condition or properties of the tissue(s) or other material(s) with which the interrogating signal interacts. In particular, interrogating ultrasonic signals are often transmitted into an organism and the received signal is due to backscattering or echos of the interrogating signal by one or more tissue types. Even more particularly, an ultrasonic interrogating signal is often transmitted into a region of

25   the organism where blood flow is expected and the received signal is a modified version of the interrogating signal wherein such modifications are due in whole or part to frequency shifts caused by Doppler effects caused by movement of tissues, including movement of blood. In this case it is desirable to measure and process the received signal in a manner such that the amount of Doppler

frequency shift that occurs can be ascertained in order to estimate the various velocities of the

moving tissue, particularly the velocities of the flowing blood.

Ultrasonic Doppler techniques typically entail using a hand-held probe that contains one or

more transducers that are used to transmit the interrogating signal and to produce a response,

5   typically in the form of a time-varying voltage, to the received signal. The interrogating ultrasonic

signal typically has a frequency in the radio frequency (RF) range of about 0.5 million Hertz (0.5

MHz) to about 20 MHz, although other frequencies may be used. The interrogating signal may be

sent continuously, in which case the method employed is termed "continuous wave" (CW) or the

interrogating signal may be sent intermittently, usually on a periodic basis, in which case the method

10   employed is termed "pulsed." When pulsed Doppler is used it is possible to interrogate tissue over

a range of specified distances from the transducers by measuring the received signal only for a

selected duration that occurs after a predetermined time delay after the pulse is sent. If the speed of

sound for the intervening tissue is known then the time delay converts directly into a depth of tissue

being interrogated. Interrogating a specified tissue depth often provides clinically valuable

15   information that cannot be obtained if CW Doppler is used. CW Doppler systems are often simpler

than pulsed Doppler systems, although they cannot provide information that is known to come from

tissue associated with a specific time delay (and corresponding estimated tissue depth).

Regardless of whether the Doppler system is pulsed or continuous, the spectral content of

the received signal is analyzed to assess how the tissue modified the interrogating signal. The

20   spectral frequency content of the signal is a measure of the amplitudes of each of the constituent

frequencies that, when combined, form the total signal. The spectral content is typically represented

as either the amplitude of each of the frequencies or as the power of each of the frequencies. When

the spectral content data are graphed the resulting graphs have a variety of names including

spectrograms and spectral density functions. The term "spectral density function" (SD function) will

25   be used as the general term below. When the frequency content of the interrogating signal is known

it can be compared to the spectrum of the received signal and the differences between the spectra

can be interpreted as arising from the velocity profiles of the tissues through which the signals

passed. The velocity profiles can be back-calculated from spectral differences using established

methods that are well known.

To date, a variety of methods have been used to estimate the spectral frequency content of the received signals. The main spectral analysis methods are traditionally classified as being either nonparametric or parametric methods.

The nonparametric methods are related to using Fourier series and they estimate the spectral content at equally spaced frequency points. The extant methods include Fourier and z-transform (including the Chirp z-transform). For brevity, all methods which employ assessing spectral content at equally spaced frequency points are referred to as Fourier methods.

Fourier methods, typically in the form of fast Fourier transforms (FFT), are well known and decompose a signal into frequencies that are integral multiples of a fundamental frequency. The integral multiple frequencies form an orthogonal basis in the frequency domain. This information can be used to approximate the signal as a trigonometric polynomial, as is well known. Fourier methods have been widely used to analyze the received signal because such methods have been refined to the point where they are fast and have well defined properties.

The other class of spectral analysis methods are parametric methods. Parametric methods use time-series analysis to estimate the parameters for a rational function model. The rational function can have either only poles, which is an autoregressive (AR) model, only zeros, which is a moving average (MA) model, or both poles and zeros, which is an autoregressive-moving average (ARMA) model. Included in these methods are maximum entropy methods (MEM). The number of model parameters used in a parametric method is termed the order of the model. The various methods that exist for calculating the values of the model parameters, such as the Burg algorithm and Durbin's first and second methods, are well known. An advantage of parametric methods is that they can produce reasonable spectral estimates using very few cycles or fractions of cycles of data.

SUMMARY OF THE INVENTION

It has been recognized that alternatespectral analysis methods including alternate time-to-frequency domain transforms may be beneficial for a variety of signal processing applications, including certain medical applications. In particular, in spite of the extensive use of Fourier methods, they suffer from significant deficiencies. Parametric methods also suffer significant deficiencies. With regard to Fourier methods, the most significant deficiency is that these methods produce

spectral estimates for only a small number of discrete frequencies, even when it is expected or known that the real spectrum covers a continuous range of frequencies or that the spectrum may contain frequencies different from those that form the basis that is the output of the Fourier transform. For example, the signal may contain frequencies that are not an integral multiple of the

5   fundamental frequency. When the basis is not correct, either because the underlying spectrum is continuous or because it contains frequencies other than those in the basis of the output of the Fourier transform, significant errors are produced. These errors manifest themselves as a spreading of the spectrum in which amplitude (or power) is spread out over frequencies that are not actually in the signal. Fourier methods approximate the actual frequency content of the signal by spreading

10   amplitudes to frequencies away from the basis frequencies. Such spreading is a well-known problem with Fourier methods and is called spectral broadening. Spectral broadening cannot be eliminated, although various methods have been employed to reduce it such as those that rely on windows. These windowing methods are not relevant to the current invention other than their well known existence and extensive use confirms that spectral broadening is a significant problem caused

15   by the discrete approximations that are inherent in Fourier methods. Spectral broadening will always occur when discrete frequencies are used to estimate the spectral content of signals that have spectra that are either continuous or composed of such a multitude of closely spaced frequencies that they are essentially continuous, such as the signals generated by Doppler shifting of an interrogating signal from blood flowing in an artery.

20   Another major problem of Fourier methods is that they cannot be used to examine all frequencies of interest. They can only include frequencies that are harmonics of the fundamental frequency used in the measurement. These problems, again, are well known and they limit the ability of Fourier methods to examine frequencies that are lower than the fundamental frequency used in the measurement. Furthermore, this limitation leads to the frequency resolution equaling the

25   fundamental frequency. For example, to estimate the amplitudes of frequencies spaced 20 Hz apart the fundamental frequency must be 20 Hz or lower. Therefore, if an interrogating signal has a frequency of 2 MHz and the need exists to discriminate the received signal with a resolution of 20 Hz than at least 100,000 harmonics are in the basis. This large number of harmonics creates significant computational difficulties and is generally regarded as impractical for a variety of reasons,

30   including the requirement to use many data points to perform such calculations.

The limitation that all basis frequencies for Fourier series are based on a fundamental frequency also causes other problems, such as when the received signal is not stationary (varies with an independent variable, such as time). This situation will exist, for example, when the received signal has its spectral content changing as the result of the interrogating signal interacting with blood

5   flowing through an artery. The velocity profile changes in an artery as the blood pressure varies in response to the beating of the heart and the variations in the blood's velocity profile will lead to a nonstationary signal that has a time-varying SD function. If the SD function is to be measured over a short time interval, such as over 0.01 second, then the fundamental frequency for the Fourier analysis cannot provide a better frequency resolution than that which is associated with the

10   measurement time interval. For example, a data sample with a period of 0.01 second will have a fundamental frequency of 100 Hz, which precludes calculating the spectral content of signals with a frequency resolution better than 100 Hz .

       Parametric methods also suffer deficiencies that are significant for certain applications. In particular, these methods minimize the sum of the squared errors between the measured data and

15   the values estimated by the model. As described later, measured data are necessarily digitized and this process introduces errors that produce a wide variety of equally valid models. Minimizing least squared errors does not account for the possibility of such alternative solutions.

       Another shortcoming of parametric methods is their critical dependence on the order of the model chosen. For example, in the most common model, AR models, if too few poles are used

20   then spectral peaks are not reliably detected. If too many poles are selected then false peaks are introduced. Proper model selection is nontrivial and requires a computationally intensive search process that involves calculating models using a range of orders and then selecting among them.

       It has also been recognized that digitization errors can significantly affect spectral analyses in certain applications. The input analog signal is not used directly to calculate spectral density

25   functions. First, the measured data are digitized and given discrete values. This process quantizes the data and introduces errors. These digitization errors (also called discretization errors and quantization errors) confound spectral analysis. Each digitized measurement is an estimate of the actual value and has an upper and lower error limit, within which any value is equally likely to occur. Even though any value within the error limits is equally likely to be the true value, the digitized value

30   is conventionally established as the midpoint of the range between the lower and upper error limits.

Any value within the range is equally likely, leading to an infinite number of possible values that can be used to calculate spectra. Thus an infinite number of candidate spectral density functions exist.

5   Existing spectral analysis methods do not explicitly account for this range of possible solutions. These methods can be interpreted as using least squared errors or other minimum norm methods that attempt to minimize differences between predicted values and measured values. The structure of these problems makes them sensitive to such errors when they are solved using minimum norm methods, and in particular when solved using least squared errors methods and produce an ill-posed problem. These ill posed problems are well known and regularization

10   techniques are used to calculate reasonable estimated solutions. Regularization methods, including Tikhonov regularization (sometimes called ridge regression or Phillips regularization) are used with least squared errors fits and augment the least squared errors model with an auxiliary function that is extremized (such as minimized). Regularization methods produce stable results, but trade off quality of the fitted solution (comparison of the predicted values to the measured data) with some measure

15   of the regularization function. Making this tradeoff requires using a weighting factor, which is also called the regularization parameter, that trades off the weight given to fitting the data with the weight given to the regularization function.

     Existing methods for solving ill-posed problems have several deficiencies. They require calculating the weighting factor, which is difficult and can have a significant effect on the results.

20   They fail to explicitly recognize that a solution is only valid when all of the estimated values are within the digitization error bins of the measured values. This deficiency manifests itself in solutions that have estimated values that are outside of the digitization bin for one or more measured values, and, thus, are not correct. Another deficiency is that in practice the methods rely on using minimizing least squared errors, which tends to force estimated values toward the center of the

25   digitization bins, thus not providing a monotonically convergent means for calculating viable solutions by adjusting the regularization parameter.

     Therefore, significant deficiencies exist in the prior art regarding the spectral analysis of time-varying signals and signals when it is desired to obtain fine frequency resolution.

     The present invention overcomes problems associated with determining spectral content

30   from signals where the fundamental frequency of the signal being processed corresponds to a time

period longer than what has been measured and/or the underlying spectrum is, at least over a range of interest, continuous or substantially continuous (occasionally hereinafter collectively referenced as "substantially continuous"). The present invention further enables spectral analysis where the signal is nonstationary and where the signal has digitization errors.

5   More specifically, the present invention is directed to a method and apparatus ("utility") involving a time-to-frequency domain transform to define a substantially continuous spectrum or other spectrum including non-zero values at irregularly spaced frequency intervals. The invention thus allows for spectral estimates for more than a small number of discrete frequencies thereby enabling a more accurate representation of certain real spectra and reducing or eliminating spectral

10   broadening. The invention also allows for examination of a greater range of frequencies of interest and provides frequency resolutions that are not limited to a fundamental frequency. In this regard, the invention allows for determination of a spectrum based on analysis of a portion of the time-based signal that is shorter than a full period of a component of the time-based signal. Moreover, the present invention allows for enhanced analysis of received signals that are not stationary, i.e.,

15   that vary with an independent variable such as time. Therefore, the invention has particular advantages with respect to medical contexts where the received signal has a changing spectral content as a result of interaction of an interrogating signal with moving physiological material such as blood flowing through an artery. The invention also enables reduction in digitization or quantization errors associated with analog-to-digital conversion.

20   According to one aspect of the present invention, a utility is provided for processing an input signal, such as a medical diagnostic signal, to obtain a spectrum that includes non-zero values at irregularly spaced intervals. An associated process involves receiving time-based information corresponding to one or more predetermined time intervals of a time-based signal, performing a transform on the time-based information to obtain a frequency spectrum including non-zero values

25   at irregularly spaced intervals, and operating a processor in a signal processing environment for using the transform to provide an output based on the time-based signal. The spectrum includes at least one pair of successive points (i.e. non-zero values) that has a different frequency spacing than at least one other set of successive points. Preferably, the spectrum includes a set of non-zero values including a first nonzero value at a first frequency and a second non-zero value at a second

frequency greater than the first frequency where the second frequency is a non-integer multiple of each frequency of the set of frequencies other than the second frequency.

According to another aspect of the present invention, a utility is provided for processing an input signal, such as a medical diagnostic signal, so as to generate a spectrum based on information corresponding to a short time interval of the input signal. A corresponding method includes the steps of receiving time-based information corresponding to one or more defined time intervals of the time-based signal, where the time-based signal includes a component having a period that is longer than the time interval, performing a transform on the time-based information to obtain a frequency spectrum for the time-based signal, and operating a signal processor in a signal processing environment for using the transform to provide an output based on a time-based signal. This process can be used to obtain a frequency spectrum where the received time-based information corresponds to a time interval that is less than half of the period of the signal component.

According to a further aspect of the present invention, a utility is provided for processing an input signal to obtain a substantially continuous spectrum. An associated method includes the steps of receiving time-based information corresponding to one or more defined time intervals of a time-based signal, performing a transform on the time-based information to obtain a frequency spectrum for the time-based signal where the spectrum defines a substantially continuous function across a frequency range that has non-zero values for a majority of frequencies in the range, and operating a processor in a signal processing environment for using the transform to provide an output based on the time-based signal. Such a substantially continuous spectrum can more accurately characterize the real spectrum under analysis, thereby avoiding spectral broadening and other shortcomings of certain prior art processes.

According to a still further aspect of the present invention, a utility is provided for processing an input signal so as to account for a digitization or quantization error associated with analog to digital conversion. An associated method includes the steps of receiving time-based information corresponding to one or more defined time intervals of a time-based signal, where the time-based signal is an analog signal and the time-based information is digital information, performing a transform on the time-based information to obtain a frequency spectrum for the time-based signal, where the step of performing a transform involves accounting for a digitization error associated with a difference between the analog time-based signal and the digital time-based

8

information, and operating a processor in a signal processing environment for using the transform to provide an output based on the time-based signal. The invention thus allows for accurate definition of a spectrum based on the input signal for certain cases where accurately defining such a spectrum may be difficult or impossible due to digitization error.

5   In accordance with another aspect of the present invention, a utility is provided for use in connection with converting a signal from an analog form to a digital form. The utility involves receiving an analog signal, defining a number of digitization ranges or bins, and assigning a specific value within a bin to a sample related to the analog signal based on a mathematical model where the specific value can vary relative to the boundaries of the bin. Preferably, the mathematical model

10   determines the specific value by solving a constrained optimization problem involving an objective function having one or more constraints. For example, such constraints may relate to one or more limit values of a bin, a nonnegativity constraint, and a peak count within the bin. For example, the objective function may be constrained to be convex and positive within the bin range with slopes determined by predefined parameters at or near the bin limits. The constraints may be implemented

15   as penalty or barrier functions, for example, Heaviside functions. In one implementation such a function approximates a Heaviside function and is tapered at an area corresponding to a constraint value (e.g., a bin limit value) such that the function is free from singularities (undefined function values) at that area.

          In one implementation, the invention is used to determine dimension-related information regarding a flow channel of a physiological fluid based on analysis of a signal modulated based on

20   interaction with the fluid of the flow stream. A corresponding system includes a detector for receiving a signal from the fluid and providing an analog electronic signal based on the received signal, an analog-to-digital convertor for providing a digital signal based on the analog electronic signal, a processing means (processor) for first processing the digital signal to obtain spectrum

25   information corresponding to a frequency spectrum including non-zero values that are not at regularly spaced frequencies, and a second processing of the spectrum information to obtain dimension-related information for the flow channel. The dimension-related information may be, for example, a channel dimension or a volumetric flow rate of the fluid.

30   BRIEF DESCRIPTION OF THE FIGURES

For a more complete understanding of the present invention and further advantages thereof, reference is now made to the following Detailed Description taken in conjunction with the figures as described below:

DETAILED DESCRIPTION

The interrogating signal emitted from an energy emitter, such as a transducer, can take on many forms and EQ. 1 is a general representation of an interrogating signal with frequency f.

$$f(t) = E(t) \sin(2\pi f t) \quad (1)$$

$E(t)$ is a function that describes how the amplitude varies with time. For example, if the emitted signal is a pulse then $E(t)$ can be a function that includes a phase shift term that will

periodically repeat with period τ. An example of a signal that has such a form is a Gaussian pulse. The amplitude of a Gaussian pulse will vary with time as defined by EQ. 2

$$E(t) = \frac{1}{2} \frac{\sqrt{2} \, e^{\left(-1/2 \frac{(t-\tau)^2 f^2}{b^2} \right)}}{\sqrt{\pi}} \quad (2)$$

A Gaussian pulse will vary with time as defined by EQ. 3 and a single Gaussian pulse is illustrated in FIG. 2.

$$f(t) = \frac{1}{2} \frac{\sqrt{2} \, e^{\left(-1/2 \frac{(t-\tau)^2 f^2}{b^2} \right)}}{\sqrt{\pi}} \sin(2\pi f t) \quad (3)$$

These equations can be used with either pulsed or continuous wave systems. In the context of ultrasound medical analysis, the pulsed wave has the advantage of allowing selection of the tissue depth to interrogate, which can reduce errors caused by Doppler shifts caused by blood deeper than the region of interest, such as the ascending aorta. The pulsed wave approach leads to pulses that have a transient response, and this needs to be dealt with, possibly by waiting for the transient to pass out of range before collecting data. Another approach could be to model the transient explicitly as a pure tone that has changing amplitude or as a pure tone that has multiple amplitudes that phase shift with time. Using a continuous wave avoids the transient response issues and leads to a pure tone being used to interrogate the tissue. The continuous wave approach, however, has the confounding factor of receiving Doppler-shifted signals from regions other than the selected depth of the ascending aorta, or whatever other region is of particular interest. Depth dependent signals could be inferred from signals that are created from different emitted frequencies (these would need to be much different, such as 1 MHz and 3 MHz) such that attenuation effects are substantially different. The signal strength differences could be matched against spectra to start getting depth-related data. Higher frequencies attenuate more rapidly than lower ones so using higher frequencies reduces the interference from deeper tissues, but at the expense of reducing the signal from the region of interest.

As described earlier, the interrogating signal can be used to energize tissue. The resulting backscattered signal can be sensed and analyzed to determine useful properties of the tissue that

backscattered the signal. Such a signal can comprise, for example, the received signal that echos

back from a multitude of reflectors or backscatterers moving at one or more velocities that have

been insonified by an ultrasonic interrogating signal. Among the purposes for which the present

invention can be used is determining the SD functions of backscattered signals from interrogating

5   signals that have either constant or time-varying amplitudes, including time varying signals that

include one or more pulses, repeating sets of pulses, or any other amplitude function that varies

with an independent variable. As discussed above, time is an example of such an independent

variable.

For illustrative purposes, the following description and derivations will use as an example

10   signals such as those that may occur when ultrasonic signals are used to interrogate tissues in a living

organism. This illustration is not limiting in that the present invention does not require using

interrogating signals or a response from such a signal. The measured data used by the present

invention may be acquired using any suitable means that eventually produces two or more digitized

data, including transducers followed by subsequent processing or data input or read from one or

15   more sources of data that may have been previously stored, such as tables or graphs stored as hard

copy or using electronic, magnetic, or optical means.

Let $A(t,f)$ be the SD function (in this case the function is the same as an amplitude density

function). The general form of the signal for which the SD function is to be obtained is EQ. 4.

$$g(t) = \int_{f_L}^{f_H} A(t,f)\sin(2\pi f t + \phi(f))\, df \quad (4)$$

20   As described in more detail below, the present invention determines SD functions for

processes with two or more independent variables, such as signals that vary with time, $g(t)$, when

the SD function depends on at least one independent variable, f, that has one or more nonzero

values over one or more parts of the interval $[f_L, f_H]$, where f_L and f_H are the lower and upper limits

of f. EQ. 4 defines an amplitude function, $A(t,f)$, that varies based on the values of two

25   independent variables, t and f. For this example, the variable t represents time and the variable f

represents frequency. The $A(t,f)$ function can be separated into functions of f and of t.

$$A(t,f) = E(t)\,B(f) \quad (5)$$

When EQ. 5 is substituted into EQ. 4 the result is a general expression for the time varying signal $g(t)$ for which the SD function $B(f)$ is sought.

$$g(t) = \int_{f_L}^{f_H} E(t) \, B(f) \sin(2\pi f t + \phi(f)) \, df \quad (6)$$

EQ. 6 can be rearranged to become EQ. 7

$$g(t) = \int_{f_L}^{f_H} E(t) \, B(f) \, (\sin(2\pi f t) \cos(\phi(f)) + \cos(2\pi f t) \sin(\phi(f))) \, df \quad (7)$$

5   The following terms are defined for coefficients.

$$F(f) = B(f) \sin(\phi(f)) \quad (8)$$
$$G(f) = B(f) \cos(\phi(f)) \quad (9)$$

Substituting these equations for the coefficients into EQ. 7 leads to EQ. 10

$$g(t) = E(t) \int_{f_L}^{f_H} F(f) \cos(2\pi f t) + G(f) \sin(2\pi f t) \, df \quad (10)$$

10   Distributing the integral over the sum of the sine and cosine terms leads to EQ. 11.

$$g(t) = E(t) \left(\int_{f_L}^{f_H} F(f) \cos(2\pi f t) \, df + \int_{f_L}^{f_H} G(f) \sin(2\pi f t) \, df \right) \quad (11)$$

Define spectral density component functions (SDC functions) as functions from which the SD function can be calculated. $F(f)$ and $G(f)$ are SDC functions. As is well known, $F(f)$ and $G(f)$

15   can be used to calculate the spectral density function B(f) using EQ. 12.

$$B(f) = \sqrt{F(f)^2 + G(f)^2} \quad (12)$$

The amplitude functions can also be used to calculate the phase shift for each frequency using EQ. 13.

$$\phi(f) = \arctan\left(\frac{G(f)}{F(f)}\right) \quad (13)$$

20   SD functions and SDC functions both characterize how the amplitude of a function varies with respect to an independent variable and are collectively referred to as amplitude functions. For example, $F(f)$ characterizes how the amplitude of $F(f) \cos(2\pi f t)$ varies with frequency. Either the SD function itself or SDC functions are to be estimated using the means described below.

15

The SD function or the SDC functions, such as $F(f)$ and $G(f)$, may be represented using various functions. The functions used to represent amplitude functions will generally have parameters that, when selected correctly, will lead to useful estimated values that are calculated for $g(t)$. Amplitude functions may be represented by any functional form for which parameters can be

5   determined that produce estimated values for $g(t)$ that differ from the actual values of $g(t)$ in a manner that meets predetermined objectives. An example of such a predetermined objective is having the errors between the measured values and estimated values of $g(t)$ being small enough that the estimated values for $g(t)$ provide useful approximations of the actual values of $g(t)$. Useful values for $g(t)$ occur when the parameters specified for the functions that represent the SDC

10   functions can be used to calculate the SD function of $g(t)$.

One aspect of the present invention involves structure and methodology for using values of $g(t)$ and $E(t)$, either based on measurements, simulations, or other suitable means, to determine the specific values of the parameters that characterize the SD function or one or more SDC functions when such amplitude functions are composed of one or more substantially continuous functions or

15   when such amplitude functions use values for one or more independent variables, such as frequency, that lead to non-orthogonal basis functions constituting $g(t)$. Using continuous functions to represent one or more amplitude functions over at least part of the domain of an independent variable (such as time) is particularly advantageous when a multitude of signals with various frequencies combine to produce a composite signal that is the measured signal $g(t)$. Such a composite signal occurs, for

20   example, when an interrogating signal backscatters from the multitude of particles traveling over a wide range of velocities as they are being carried in a flowing fluid. An example is the signal produced when an interrogating ultrasonic signal backscatters from the blood cells traveling through a living organism's aorta.

In contrast to the present invention, Fourier methods represent the amplitude functions using

25   discrete values of frequencies that are selected as integer multiples of a fundamental frequency. The frequencies used in Fourier methods are carefully selected such that the resulting basis functions are both orthogonal and normal (orthonormal). As described earlier, using discrete frequencies, particularly when they are restricted to being harmonics of a fundamental frequency, to approximate continuous functions leads to significant inaccuracies, such as spectral broadening. The present

30   invention improves upon Fourier methods by using frequencies that can be selected substantially

16

without limitation as to which frequencies are selected and also by allowing the use of continuous functions to represent continuous amplitude functions.

In the present invention, one or more functions are selected to represent the SD or SDC functions. The function used to represent an SDC, such as $F(f)$, may be zero. The functions used
5   to represent the amplitude functions may be discrete, such as one or more vectors of discrete frequencies starting at any frequency of interest and spaced at any, possibly varying and nonuniform, frequencies from each other (in contrast to the evenly spaced harmonic frequencies using in Fourier methods). The functions representing amplitude functions in the present invention can be orthogonal or non-orthogonal. The functions representing amplitude functions can be either normal or not
10   normal. The function or functions used to represent amplitude functions may be one or more continuous, not continuous, or piecewise continuous functions. The functions representing amplitude functions may have one or more intersecting, including overlapping, domains. An overlapping domain between two or more functions is an interval where the domains of two or more functions intersect at more than one value. FIG. 3 illustrates an example where multiple piecewise continuous
15   functions that intersect at only their endpoints represent an amplitude function. FIG. 4 illustrates an example where multiple functions with overlapping domains model an amplitude function. One aspect of the present invention is that functions representing the amplitude functions may use one or more of the following functions: splines, B-splines, polynomials of order 1 or greater, trigonometric functions, trigonometric polynomials, exponential functions, hyperbolic functions, Heaviside
20   functions, functions that approximate Heaviside functions (discussed in detail below), or any another functions that can be used to construct one or more bases that span one or more predetermined vector spaces that can approximate or define a discrete, non-continuous, continuous, or piecewise continuous function. For example, the piecewise continuous segments illustrated in FIG. 3 are composed of quadratic polynomials and Heaviside functions. As another example, FIG. 4
25   illustrates how B-splines can combine to represent an amplitude function.

FIG. 4 also illustrates another aspect of the present invention that provides for computational flexibility. The independent and dependent variables can be offset (as illustrated in FIG. 4 where the frequency axis is offset by 2 MHz) or scaled to facilitate calculating the parameters of the functions that represent amplitude functions.

17

Using piecewise continuous functions or functions with overlapping domains is of particular

interest because parameters for function segments defined over discrete intervals can be efficiently

determined using the present invention. The union of the function segments can represent

complicated amplitude functions. FIG. 3 is a graphical example of a piecewise continuous function.

5   The points where the individual function segments of the piecewise continuous function intersect are

knot points and these are illustrated by the hollow dots in   FIG. 3.

Another aspect of the present invention involves structure and methodology employing a

calculation module that uses one or more objective measures to determine the values of the

parameters in the functions that represent the amplitude functions or SD function. One example of

10   such an objective measure is using goodness of fit. Other examples of objective measures are

minimizing the length of the amplitude functions, SD function, or SDC functions, minimizing the

squared area or absolute value of the area under the amplitude functions, SD function, or SDC

functions, and minimizing one or more measures of the slopes or changes in slopes of the amplitude

functions, SD function, or SDC functions.

15   One example of an objective measure using goodness of fit is minimizing, or approximately

minimizing, the sum of the squared errors between measured values of $g(t)$ and those predicted

using amplitude functions. Another example of an objective measure of goodness of fit is

minimizing, or approximately minimizing, the sum of squared errors when the values of one or more

of the parameters in one or more of the amplitude functions or the SD function are constrained to

20   have values with predetermined properties. Such predetermined properties may include that the

parameters are allowed to only have particular values, such as being non-negative, or that the

parameters have a particular relationship, such as the mathematical combination of two or more of

the parameters (such as one or more of their sum, difference, product, or quotient,) always be

positive or that the mathematical combination always falls within a certain range, such as being

25   approximately equal to zero. Another example of an objective measure of goodness of fit is

minimizing, or approximately minimizing, the sum of the absolute values of errors, including the

possibility that the values of one or more of the parameters in one or more of the amplitude

functions or the SD function are constrained to have values with predetermined properties. As

before, such properties include having one or more parameters being restricted to particular values,

30   such as being non-negative, or that the parameters have a particular relationship, such as the sum of

18

two or more parameters always be non-negative or that they approximately equal zero. These aspects of the invention are covered in more detail below.

Referring to EQ. 11, in accordance with a further aspect of the present invention, optimized results can beobtained by setting the values for f_L and f_H respectively low enough and high enough that they cover the full range of frequencies that constitute the spectrum of $g(t)$. The values for f_L and f_H may be determined concurrently with determining the values of the parameters for the functions representing the SDC functions, such as $G(f)$ and $F(f)$, or they may be selected prior to determining the parameters for the functions representing $G(f)$ and $F(f)$. Selecting f_L and f_H ahead of time is often possible. For example, the conditions that produce $g(t)$ are generally known. A predetermined lowest frequency may be based on the frequency of an interrogating signal interacting with one or more moving media. For example, the lowest frequency can be predetermined based on the frequency of the interrogating signal and the lowest frequency expected from the moving media after including factors such as measurement errors or noise that can affect the frequencies produced by the interaction of the interrogating signals with the moving media.

Another aspect of the invention involves using single continuous functions that span the entire domain of frequencies as amplitude functions or as SD functions. An example of such a function is EQ. 14. This equation characterizes the amplitude functions that occur during laminar fluid flow with measurement noise. This equation would become $B(f)$ in EQ. 8 and   EQ. 9.

$$(14)$$

$$x(f)=\frac{1}{2}\frac{(1-F)\,e^{\left(-\frac{1}{2}\frac{\left(\frac{c\,(-f+f_e)}{f+f_e}+\mu_s\right)^2}{\sigma_s^2}\right)}}{\sqrt{\pi}\,\sigma_s}\sqrt{2}+F\int_{f_e}^{\frac{(c+a)f_e}{c-a}}\frac{\left(1+\frac{c\,(-x+f_e)}{(x+f_e)\,a}\right)^{\left(2\frac{1}{k}-1\right)}e^{\left(-2\frac{c^2 f_e^2\,(-f+x)^2}{(f+f_e)^2\,(x+f_e)^2\,2\sigma^2}\right)}\sqrt{2}}{\sqrt{\pi}\,\sigma\,k\,a}\,dx$$

The emitted frequency sent out as the interrogating signal is f_e and the tissues move at various velocities to produce the reflected frequencies, f. The amount of tissue at each velocity produces the relative amplitudes. The parameters are:

1. F the fraction of the insonified region that has flowing liquid.

2. k: the shape exponent for the flow region

3. σ_s : the standard deviation for the velocity in the slow/nonflowing region's reflectors.

4. f_e : the frequency of the interrogating signal.

5. a: the maximum value for velocity that is achieved at the time that a measurement is made.

6. μ_s : the mean velocity of the slow moving/nonflowing region's reflectors.

7. σ : the standard deviation of the measurement error associated with the flowing fluid.

8. c: the speed of sound.

A further aspect of the present invention involves using such a single continuous amplitude function in combination with a function that represents the phase shift, $\phi(f)$, see, e.g., EQ. 8 and EQ. 9, to produce single continuous functions that represent amplitude functions that span the entire domain of frequencies. An additional aspect of the present invention involves using such a single continuous amplitude function, e.g., in combination with functions that represent the $sin(\phi(f))$ and $cos(\phi(f))$ in EQ. 8 and EQ. 9 to produce single SDC functions that span the entire domain of frequencies. A further aspect of the present invention relates to representing $\phi(f)$, $sin(\phi(f))$, or $cos(\phi(f))$ using either constants or polynomials with degree of 1 or greater. A further aspect of the present invention involves selecting the frequency of the interrogating signal used with Doppler measurements of velocities such that $\phi(f)$, $sin(\phi(f))$, or $cos(\phi(f))$ may be represented using either constants or a linear function.

A further aspect of the present invention involves characterizing the phase shift when using Doppler measurements of fluid flows using EQ. 15 to represent phase angle.

$$\phi = -\arctan\left(\frac{1}{2}\frac{4f\pi\sin\left(\frac{fD\pi}{c}\right)\cos\left(\frac{fD\pi}{c}\right) - 2f^x c b \cos\left(\frac{fD\pi}{c}\right)^2 + f^x c b + f^x e^{(bf^x D)} b c}{2f\pi\cos\left(\frac{fD\pi}{c}\right)^2 - f\pi + c bf^x \sin\left(\frac{fD\pi}{c}\right)\cos\left(\frac{fD\pi}{c}\right) - f\pi e^{(bf^x D)}}\right) \tag{15}$$

where

ϕ = phase angle in radians

f = frequency in Hz

D = depth of region from which Doppler-shifted signals are being generated

c = speed of signal, such as speed of sound

x = attenuation exponent

b = attenuation constant

5   For the case where x=1 EQ. 15 simplifies to become EQ. 16

$$\phi = \arctan\left(\frac{-2\,\pi\,\sin\left(2\,\frac{f D\,\pi}{c}\right) + c\,b\,\cos\left(2\,\frac{f D\,\pi}{c}\right) - e^{(bfD)}\,b\,c}{2\,\pi\,\cos\left(2\,\frac{f D\,\pi}{c}\right) + c\,b\,\sin\left(2\,\frac{f D\,\pi}{c}\right) - 2\,\pi\,e^{(bfD)}}\right) \quad (16)$$

The attenuation constant, b, can be calculated from the attenuation factor using EQ. 17

$$b = .5000000000 \quad 10^{-7}\,\beta\,\ln(10) \quad (17)$$

10   where β is the attenuation factor in dB.

FIG. 5 illustrates how the phase angle can change with frequency. As is known, selecting the emitted frequency determines the frequencies of the returned signal. As previously mentioned, one aspect of the present invention involves selecting the emitted frequency such that the returned signals are confined to an approximately linear region of the phase angle response. For example, in

15   FIG. 5 selecting an emitted frequency that leads to return frequencies around 2.079 MHz would be in a region where phase angle varies close to linearly with frequency, whereas selecting an emitted frequency that leads to return frequencies about 2.17 MHz would be in a region where phase angle does not vary linearly with frequency. More specifically, if 2.078 MHz is selected as the emitted frequency and blood flow in the ascending aorta of an adult human is being measured using an

20   interrogating signal beamed down the axis of the aorta, then frequencies in the range of approximately 2.078 to 2.093 MHz will be returned. FIG. 6 illustrates an example phase angle response over this frequency range. In this example, selecting 2.078 MHz leads to phase angle responses that may be modeled as a linear function.

FIG. 5 illustrates that half of the phase angle response will be over frequencies that lead to

25   positive phase angles. Selecting an emitted frequency such that the phase angle response is always positive is beneficial. When the phase angles are always positive the amplitude response function $F(f)$ of EQ. 1 will always be positive. If the phase angle becomes negative the $sin(\phi(f))$ term will

be negative, leading to negative values over at least part of the range of $F(f)$. Having only positive

phase angles avoids this condition. Calculating correct values for $F(f)$ and $G(f)$ is made easier

when it is known that $F(f)$ and $G(f)$ are always positive. Selecting the emitted frequency for

Doppler velocity measurements such that the phase angle response is always positive is included in

5   accordance with another aspect of the present invention.

Another aspect of the present invention involves selecting the emitted frequency so that the

phase angle response is always positive and in a region where the phase response is approximately

linear. FIG. 7 illustrates an example in accordance with this aspect of the present invention.

A further aspect of the present invention involves selection of the emitted frequency so that

10   at least a portion of the phase angle response is close to zero.

FIG. 8 and FIG. 9 illustrate how thus selecting the emitted frequency yields $sin(\phi(f))$ and

$cos(\phi(f))$ functions that can be approximated. $sin(\phi(f))$ can be approximated in many cases using

a linear function. The range of $cos(\phi(f))$ is small. Depending upon the precision needed it may be

approximated as a constant (such as 0.98 in this example), as a linear function, or as a higher order

15   polynomial. A linear approximation would pick up most of the variation and be adequate for many

applications.

Compare the results illustrated in FIGS. 7, 8 and 9 with what happens if the emitted

frequency is not selected according to this aspect of the present invention. FIGS.10, 11, and 12

illustrate the results from using an emitted frequency of 2 MHz for the conditions of this example.

20   The linear response and small domains that occurred when the emitted frequency was selected in

accordance with this aspect of the present patent are not present in FIGS.10, 11, and 12.

When the frequency distribution of a spectral distribution has been produced in a manner

that relates frequency to a calculated variable then the distribution of the calculated variable can be

25   determined. For example, if the frequency distribution has been obtained using Doppler then the

velocity at any frequency can be calculated using EQ. 18. The frequency data or data for a

calculated variable can be used to calculate the values of one or more parameters that characterize

a distribution. Such distributions are particularly useful when the methods of the present invention

are used to calculate continuous distributions. Such a distribution could be, for example a Gaussian

30   distribution that is characterized by its mean and standard deviation. EQ. 19 is an example of a

22

more complicated distribution that has six parameters. FIG. 18 is an example of such a distribution when the calculated parameter is velocity.

5

$$v = -\frac{c\,(-f+f_e)}{f+f_e} \quad (18)$$

$$x(v) = \frac{1}{2}\frac{(1-F)\,e^{\left(-1/2\frac{(-v+\mu_s)^2}{\sigma_s^2}\right)}\sqrt{2}}{\sqrt{\pi}\,\sigma_s} + F\int_0^a \frac{\left(1-\frac{x}{a}\right)^{\left(2\frac{1}{k}-1\right)}e^{\left(-1/2\frac{(v-x)^2}{\sigma^2}\right)}\sqrt{2}}{\sqrt{\pi}\,\sigma\,k\,a}\,dx \quad (19)$$

10

Other aspects of the present invention involve structure and methodology for determining the parameters of an SD function, or the SDC functions, from signals that have been distorted by digitization errors. When signals are measured and converted from analog to digital form the analog values are converted to numeric (digital) values with a limited precision. Such limited precision

15   leads to numeric values that approximate the actual values of the signal and differences between the actual values and the approximate values are errors caused by the digitization process. In particular, the present invention includes structure and methodology for calculating SD functions, or the functions that can be used to determine the SD functions, when using digitized data by constraining the calculated functions so that they have predetermined properties. Such

20   predetermined properties can include having at least one, and typically the majority or approximately all, of the values of g(t) estimated using the calculated functions falling within the ranges determined by the digitization process. Such predetermined properties can also include requiring SD functions, or the functions that can be used to determine the SD functions, having predetermined properties related to their geometries or of the parameters that define the functions.

25   For example, such methodology includes having functions that are the minimum measures of slopes or rates of changes in slopes, the lengths, or squared areas enclosed by the functions. Another example is having functions that maximize, or minimize, the magnitudes of one or more of the parameters that characterize the functions. Another example is having functions that minimize

measures of length and squared areas enclosed by functions and that maximize the magnitudes of one or more of the parameters that characterize the functions. These aspects of the invention are described more completely later.

To illustrate the aspect of the present invention that characterizes amplitude functions using multiple functions, consider the following description of a general process for estimating amplitude functions. Amplitude functions, such as those of EQ. 8 and EQ. 9, can be approximated arbitrarily closely as the sum of other functions. The following expressions define the approximations of $G(f)$ and $F(f)$ as the sums of N piecewise functions.

$$G(f) = \sum_{j=1}^{N} G_j(f) \quad (19)$$

$$F(f) = \sum_{j=1}^{N} F_j(f) \quad (20)$$

The N piecewise functions defined over each of N intervals will collectively span the domain of f that is of interest. N functions exist for $G(f)$ and N functions exist for $F(f)$, and they may or may not have overlapping domains. Each of the $G_j(f)$ and $F_j(f)$ will be defined over a domain of f with lower and upper limits. For the case of using piecewise continuous functions the union of the domains will be continuous and the individual domains will not overlap except at the points of intersection.

Substituting EQs. 19 and 20 into EQ. 11 and distributing cosine and sine terms over the sums leads to EQ. 21.

$$g(t) = E(t) \left(\int_{f_L}^{f_H} \sum_{j=1}^{N} F_j(f) \cos(2\pi f t) \, df + \int_{f_L}^{f_H} \sum_{j=1}^{N} G_j(f) \sin(2\pi f t) \, df \right) \quad (21)$$

Changing the integral of the sum to a sum of integrals leads to EQ. 22.

24

$$g(t) = E(t) \left(\left(\sum_{j=1}^{N} \int_{f_L}^{f_H} F_j(f) \cos(2\pi f t) \, df \right) + \left(\sum_{j=1}^{N} \int_{f_L}^{f_H} G_j(f) \sin(2\pi f t) \, df \right) \right) \tag{22}$$

Each of the piecewise functions is defined over part of the domain of f. Indexing the limits of integration for each of the integrals in the sums leads to EQ. 23.

$$g(t) = E(t) \left(\left(\sum_{j=1}^{N} \int_{f_{L_j}}^{f_{H_j}} F_j(f) \cos(2\pi f t) \, df \right) + \left(\sum_{j=1}^{N} \int_{f_{L_j}}^{f_{H_j}} G_j(f) \sin(2\pi f t) \, df \right) \right) \tag{23}$$

5        $E(t)$ is the envelope of the interrogating signal. If the signal being analyzed has not been modulated by an interrogating function then setting the envelope function E(t)=1 will decompose the received signal without removing the effects of the interrogating signal. If $E(t)$ is not constant then methodology in accordance with the present invention can separate the effects of $E(t)$ while decomposing the received signal into, for example, SD or SDC functions.

10

g(t) is the received signal and measured values for it are known. Dividing both sides of EQ. 23 by $E(t)$ produces EQ. 24.

$$\frac{g(t)}{E(t)} = \left(\sum_{j=1}^{N} \int_{f_{L_j}}^{f_{H_j}} F_j(f) \cos(2\pi f t) \, df \right) + \left(\sum_{j=1}^{N} \int_{f_{L_j}}^{f_{H_j}} G_j(f) \sin(2\pi f t) \, df \right) \tag{24}$$

Measured values from g(t) are the data used to calculate F(f) and G(f).

15   Use the following substitutions in EQ. 24.:

$$\frac{g(t)}{E(t)} = f(t) \tag{25}$$

$$f_{H_j} = H_j \quad (26)$$

$$f_{L_j} = L_j \quad (27)$$

and obtain EQ. 28.

$$f(t) = \left(\sum_{j=1}^{N} \int_{L_j}^{H_j} F_j(f) \cos(2 \pi f t) \, df \right) + \left(\sum_{j=1}^{N} \int_{L_j}^{H_j} G_j(f) \sin(2 \pi f t) \, df \right) \quad (28)$$

5   As stated earlier, the piecewise continuous functions may be polynomials, such as linear or quadratic functions. Each of the function segments will have its coefficients, such as the slope and intercept for linear functions. For illustration of the principles of the present invention, the following example uses quadratic functions and derives expressions that may be used to solve for the coefficients that specify each of the quadratic function segments.

10   Let the following expressions specify the piecewise quadratic function segments:

$$G_j(f) = a_j f^2 + b_j f + c_j \quad (29)$$

$$F_j(f) = x_j f^2 + y_j f + z_j \quad (30)$$

Substituting these expressions into EQ. 28 leads to EQ. 31.

$$f(t) = \left(\sum_{j=1}^{N} \int_{L_j}^{H_j} (x_j f^2 + y_j f + z_j) \cos(2 \pi f t) \, df \right) + \left(\sum_{j=1}^{N} \int_{L_j}^{H_j} (a_j f^2 + b_j f + c_j) \sin(2 \pi f t) \, df \right) \quad (31)$$

15   This equation is linear in the coefficients x_j, y_j, z_j, a_j, b_j, and c_j, and the integrals have analytical solutions. FIG. 14 has the analytical solution. This equation is a general expression. No restrictions have been placed on the interval limits $[L_j, H_j]$ so they do not need to be continuous and they could overlap. Therefore, the function segments could be discontinuous. No constraint has been placed on the slopes of function segments matching if function segments are continuous. The

20   current formulation, without constraints, is broadly applicable.

26

The unknowns to be solved for are the coefficients x_j, y_j, z_j, a_j, b_j, and c_j for $j=1$ .. N. Represent these unknowns as the vector $|d|$. Once values have been estimated for $|d|$ then estimated values for $f(t)$ can be calculated using EQ. 31. The values of $|d|$ can be solved for using one or more objective criteria, such as the estimated values for $f(t)$ having the least sum of squared errors compared to the measured values for $f(t)$. EQ. 31 is linear in x_j, y_j, z_j, a_j, b_j, and c_j so minimum norm methods, such as least squared errors, may be employed to solve for the unknowns. The number of frequency intervals and the lower and upper frequencies for each function segment are values that are selected, and thus, are known. Similarly the times when the measurements are taken are known. These values are inserted into the equation in FIG. 14 and the result is a set of coefficients that multiply each of the x_j, y_j, z_j, a_j, b_j, and c_j. This set of coefficients forms a rectangular matrix $\|A\|$. Designate the vector of measured values of f(t) as $|f|$. The resulting matrix equation is EQ. 32.

$$\|A\| \, |d| = |f| \qquad\qquad\qquad (32)$$

Matrix solution methods known to those skilled in the art can be used to solve for $|d|$, which contains the values for x_j, y_j, z_j, a_j, b_j, and c_j. One such method is using the Moore-Penrose generalized inverse of $\|A\|$. Designating $\|P\|$ as the Moore-Penrose generalized inverse of $\|A\|$ leads to EQ. 33. Singular value decomposition is another method that can be used.

$$\|P\| \, |f| = |d| \qquad (33)$$

For the situation where the measurement times and frequency intervals are predetermined and reused multiple times only one matrix inverse needs to be calculated, which offers considerable economy of calculation. The matrix $\|A\|$ is based on the predetermined $[L_j, H_j]$ and predetermined measurement times and is calculated in advance. The Moore-Penrose generalized inverse, $\|P\|$, is also calculated in advance and stored for later use, such as in the nonvolatile memory of a computation means such as using a hard disk or other magnetic media or on a CD ROM or other optical storage media, or nonvolatile RAM or other electronic storage media. After $\|A\|$ and $\|P\|$ have been calculated and stored the values for $\|P\|$ can be retrieved and used to calculate $|d|$ whenever such calculations are desired for a vector of measured values of f(t), $|f|$ using EQ. 33, which leads immediately to values for the coefficients x_j, y_j, z_j, a_j, b_j, and c_j.

As described below, the formulation becomes substantially simpler when the lower and upper bounds for each interval, $[L_j, H_j]$ all have the same width, but they do not need to have the

27

same width or be equally spaced. No restrictions are placed on the measurement times being
equally spaced. The flexibility available for selecting the bounds for each interval and the
measurement times allows selecting nonuniform intervals, such as using Chebyshev points or
expanded Chebyshev points.

5   Methods other than the least sum of squared errors can be used to calculate the values of
the unknown coefficients . For example, the least sum of absolute value of errors (LAV) or
nonnegative least squares (NNLS) can be used. As is known to those skilled in the art, LAV can
be implemented such that constraints are placed on the values of the variables, as is also the case
with NNLS. Such constraints are particularly useful when linear function segments are used

10   because these readily lend themselves to constraints that prevent one or more of SDC functions,
such as G(f), from having any negative values.

           For the case where spectra are continuous, such as the Doppler spectra from flowing
blood, the *jth* function segment needs to connect to function segment *j-1*. Requiring continuity
means that the endpoint of the *(j-1)th* function segment must be the starting point for the *jth* function

15   segment, which leads to the lower bound of the *jth* segment being equal to the upper bound of the
(j-1)th segment. Therefore, EQ. 34 holds for piecewise continuous functions.

$$L_j = H_{j-1} \quad (34)$$

           As demonstrated below, this restriction means that the value for c_j is determined by the

values of a_{j-1}, b_{j-1}, c_{j-1} and H_j. Therefore, c_j is not an independent variable when the
20   quadratic function segments are piecewise continuous.

           Spectra from processes such as flowing blood are smooth. Smooth transitions between
quadratic function segments requires that the slope at the end of the *(j-1)th* segment match the slope
of the *jth* segment. As will be shown below, this constraint removes the set of b_j from being
independent variables.

25   The following discussion uses as examples the a_j, b_j, c_j, and the same discussion also

applies to the x_j, y_j, z_j. Include an offset for the frequency at the beginning of the interval over
which a function is being used. The equation for the *jth* function segment that goes through a

specific point is derived as follows. Designate the specific point for the lowest frequency and the

amplitude at the lowest frequency by using the ordered pair (L_j, β_j). β_j is the beginning amplitude

for the *jth* function segment. The *jth* function segment has its own coefficients a_j, b_j, and intercept

c_j.

5   The starting point for the *jth* function segment has the same coordinates as the endpoint for

the *(j-1)th* function segment. Designate the endpoint for the *(j-1)th* function segment as

(H_{j-1}, e_{j-1}). e_{j-1} is the ending amplitude for the *(j-1)th* function segment. The starting point for

the *jth* function segment is the endpoint of the *j-1th* function segment. The following equation gives

the amplitudes for the *jth* function segment of the SDC function when using piecewise continuous

10   quadratic functions to represent the SDC function.

$$G_j = a_j (f - H_{j-1})^2 + b_j (f - H_{j-1}) + c_j \quad (35)$$

However, $c_j = e_{j-1}$ so

$$G_j = a_j (f - H_{j-1})^2 + b_j (f - H_{j-1}) + e_{j-1} \quad (36)$$

15   For the special case when j=1 the starting point has an amplitude of 0 at the frequency f_{lb} ,

where f_{lb} is defined earlier. Clearly, a recurrence exists that gives the e_k. The basic equation is:

$$e_j = a_j (H_j - H_{j-1})^2 + b_j (H_j - H_{j-1}) + e_{j-1} \quad (37)$$

20   This equation gives the logical result that the height of the endpoint of the *jth* function

segment is the sum of the heights of all of the preceding endpoints plus the change in height that

occurs during frequency interval *j*. Using the preceding equations to eliminate c_j from EQ. 35 leads

to EQ. 38.

$$(38)$$

$$G_j = a_j (f - H_{j-1})^2 + b_j (f - H_{j-1}) + \left(\sum_{p=1}^{j-1} ((H_p - H_{p-1})^2 a_p + (H_p - H_{p-1}) b_p) \right)$$

These equations that tie the endpoints together establish continuity between the piecewise functions. The piecewise functions are now piecewise continuous. The values for the H_j are known because they are selected *a priori*. In this set of equations the only free variables are the first $N-1$ b_j and the N a_j.

Constrain the slopes of the *j-1th* and the *jth* segments to match where the segments joint. This constraint leads to the starting slope of the *jth* function segment being equal to the parameter b_j. The b_j for each segment depends only upon the values for the coefficients for the previous function segments. The end result is that each b_j depends only on the previous values of the a_j. The equation for b_j is

$$b_j = 2 \left(\sum_{q=2}^{j} (H_{q-1} - H_{q-2}) a_{q-1} \right) \quad (39)$$

For the first function segment the initial slope is zero because it starts tangent to the f-axis. Therefore, $b_1 = 0$. This situation becomes clear by examining the case where j=1.

$$G_1 = a_1 (f - H_0)^2 \quad (40)$$

Using EQ. 39 to eliminate b from EQ. 38 leads to EQ. 41 . This equation is the general expression for the *jth* function segment for piecewise continuous quadratic function segments.

$$G_j = \left(\sum_{p=1}^{j-1} (H_p - H_{p-1}) (-H_{p-1} + 2f - H_p) a_p \right) + a_j (f - H_{j-1})^2 \quad (41)$$

The endpoint of the final function segment is at frequency H_N. The SD function needs to be zero at this point and the slope also needs to be zero. These requirements also require the SDC functions, such as $G(f)$, to have zero amplitude and slope at H_N. Two boundary conditions exist

at H_N , but only one free variable, a_{N-1} , remains, therefore need to remove another degree of freedom. Set the equations for the boundary conditions that equate the slope and amplitude at the end of the *Nth* segment equal to each other (they both equal 0) and it becomes clear that a_{N-1} is not a free variable, either. Therefore, when have *N* function segments only *N-2* values of |a| need to

5   be found.

The equations become considerably simpler is equally spaced function segments are used. Assume that each of the function segments covers a frequency interval of equal width. Select a value for the lowest basis frequency, H_0 , and then index by H_δ up from it for other frequencies in the basis.

10   With equally spaced frequency intervals have

$$H_{q-1} - H_{q-2} = H_\delta \quad (41)$$

And the general expression for the *jth* quadratic function segment becomes EQ. 42.

$$(42)$$

$$G_j = H_\delta \left(\sum_{p=1}^{j-1} (-(2p-1)H_\delta + 2f - 2H_0) a_p \right) + a_j (f - H_0 - (j-1)H_\delta)^2$$

The unknown variables are the a_j. Represent these variables as the vector |a|. When

15   impose continuity and matching slopes at the knotpoints where the function segments join the values for the *(N-1)th* and *Nth* values of |a| stop being independent variables. The expressions for these variables are

$$a_{N-1} = \sum_{p=1}^{N-2} a_p (p - N) \quad (43)$$

$$a_N = -\left(\sum_{p=1}^{N-2} a_p (p - N + 1) \right) \quad (44)$$

20   The equations that give a_{N-1} and a_N confirm that the last two coefficients are completely specified by the first *N-2* coefficients and, thus, are not independent variables. These values lead to

the last two function segments working together to make the final amplitude of $G(f)$ become zero and with zero slope with smooth transitions at the knotpoints of the final two function segments.

Expressions for the $(N-1)th$ and Nth function segments in terms of the first $N-2$ elements of

$$G_{N-1} = \sum_{j=1}^{N-2} (H_\delta (-(2j-1) H_\delta + 2f - 2 H_0) a_j + (-H_0 - (N-2) H_\delta + f)^2 a_j (j-N)) \tag{45}$$

$|a|$ are

5

$$G_N = \sum_{j=1}^{N-2} (-a_j (-f + N H_\delta + H_0)^2 (j - N + 1)) \tag{46}$$

Similar relationships exist for $F(f)$, F_j and $|d|$, including

$$(47)$$

$$F_j(f) = H_\delta \left(\sum_{p=1}^{j-1} (-(2p-1)H_\delta + 2f - 2H_0) x_p \right) + (-H_0 - (j-1)H_\delta + f)^2 x_j$$

$$(48)$$

$$F_{N-1} =$$

$$\sum_{j=1}^{N-2} (H_\delta (-(2j-1)H_\delta + 2f - 2H_0) x_j + (-H_0 - (N-2)H_\delta + f)^2 x_j (j-N))$$

$$F_N = \sum_{j=1}^{N-2} (-x_j(-f + NH_\delta + H_0)^2 (j-N+1)) \quad (49)$$

Substituting the preceding equations into EQ. 22 and integrating and substituting to remove the *N-1th* and *Nth* a_j and x_j leads to EQ. 50. This equation is the final equation for the function that approximates g(t) when using equally spaced quadratic function segments that are piecewise continuous and constrained so that the slopes at the knotpoints are equal and that slopes and values

$$(50)$$

$$g(t) = E(t) \left[\sum_{j=1}^{N-2} \left(\frac{1}{8} \frac{(\sin(2\pi t(H_0 + (N-2)H_\delta))(-2N+2f) - 2(N-1-f)\sin(2\pi t(H_0 + NH_\delta)) - 2\sin(2\pi t(H_0 + fH_\delta)) + (4N-4f-2)\sin(2\pi t(H_0 + (N-1)H_\delta)) + 2\sin(2\pi t(H_0 + (f-1)H_\delta)))x_j}{\pi^3 t^3} \right. \right.$$

$$\left. \left. + \frac{\frac{1}{8}(2(N-f)\cos(2\pi t(H_0 + (N-2)H_\delta)) + 2\cos(2\pi t(H_0 + fH_\delta)) + 2(N-1-f)\cos(2\pi t(H_0 + NH_\delta)) + (-4N+4f+2)\cos(2\pi t(H_0 + (N-1)H_\delta)) - 2\cos(2\pi t(H_0 + (f-1)H_\delta)))a_j}{\pi^3 t^3} \right) \right]$$

at the beginning and end of the SD function are zero. EQ. 50 is a general formula for calculating g(t) that is linear in the variables x_j and a_j.

The times when measurements are made are known, as are the starting basis frequency, H_0 and the width of each frequency interval, H_δ. After these substitutions are made the result is a set of equations consisting of coefficients on the unknown *N-2* unknown variables a_j and the *N-2* unknown variables x_j. The coefficients can be extracted and put in a rectangular matrix $\|A\|$. $\|A\|$ will have M rows and 2(N-2) columns where:

M = number of measurements, with the measurements being t_i, i = 1 .. M

N = number of function segments (which is the same as the number of basis frequencies).

Set $\|A\|$ so that the first N–2 columns contain the coefficients for a_j and the last N–2 columns contain the coefficients for the x_j. The equation for the first N-2 columns of $\|A\|$ is EQ. 51. The equation for the last N-2 columns of $\|A\|$ is EQ. 52. These equations assume that the envelope function is known, which is often the case, such as if the interrogating function is a continuous sine or

$$A_{i,j} = \frac{1}{8} E(t_i) \, (2 \cos(2\,\pi\,(H_0 + j\,H_\delta)\,t_i) - 2 \cos(2\,\pi\,(H_0 + (j-1)\,H_\delta)\,t_i)$$ (51)
$$+ 2\,(N-j-1)\cos(2\,\pi\,t_i\,(H_0 + N\,H_\delta))$$
$$+ 2\,(N-j)\cos(2\,\pi\,t_i\,(H_0 + (N-2)\,H_\delta))$$
$$+ (-4\,N + 4\,j + 2)\cos(2\,\pi\,t_i\,(H_0 + (N-1)\,H_\delta))) \Big/ (\pi^3\,t_i^{\,3})$$

$$A_{i,j+N-2} = \frac{1}{8} E(t_i)\,(-2\,(N-j-1)\sin(2\,\pi\,t_i\,(H_0 + N\,H_\delta))$$ (52)
$$- 2\sin(2\,\pi\,(H_0 + j\,H_\delta)\,t_i) + \sin(2\,\pi\,t_i\,(H_0 + (N-2)\,H_\delta))\,(-2\,N + 2\,j)$$
$$+ (4\,N - 4\,j - 2)\sin(2\,\pi\,t_i\,(H_0 + (N-1)\,H_\delta))$$
$$+ 2\sin(2\,\pi\,(H_0 + (j-1)\,H_\delta)\,t_i)) \Big/ (\pi^3\,t_i^{\,3})$$

5 has other known characteristics. If the signal being analyzed has not been modulated by an interrogating function then set the envelope function $E(t)=1$ and the method will decompose the complete signal and not attempt to remove the effects of the interrogating signal.

Designate the values of the (currently unknown) N-2 values of a_j as $|a|$ and designate the (currently unknown) N-2 values of x_j as $|x|$. Designate the vector $|d|$ as being $|a|$ stacked over $|x|$,

10 thus $|d|$ contains the unknown variables a_j, $j = 1 .. N$-2 followed by the unknown variables x_j, $j = .. N$-2. The values $|d|$ can solved as described earlier, using, for example, the Moore-Penrose generalized inverse.

FIG. 15 through FIG. 23 illustrate the method just described using an example. The example uses a small number of measurements, 20, and seven quadratic function segments. FIG.

15 15 lists the parameters used to calculate the $\|A\|$ matrix shown in FIG. 16. The Moore-Penrose generalized inverse of $\|A\|$ is shown in FIG. 17. Only $\|P\|$ is needed to calculate the values of $|d|$ from measured values of f(t). The 20 values of g(t) used in the example were calculated using the G(f) and F(f) shown in FIG. 19. A successful calculation will produce estimated G(f) and F(f) close